Warmup from last time:

Contour diagrams for ① $z = 2x^2 - 8y^2$
② $z = xy$

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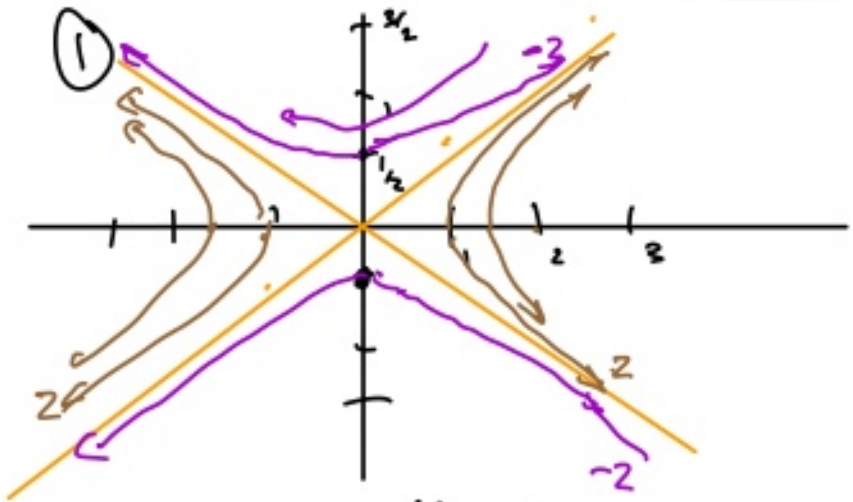
①



$2x^2 - 8y^2 = -2, -1, 0, 1, 2$

$2x^2 - 8y^2 = -2$
$-x^2 + 4y^2 = 1$

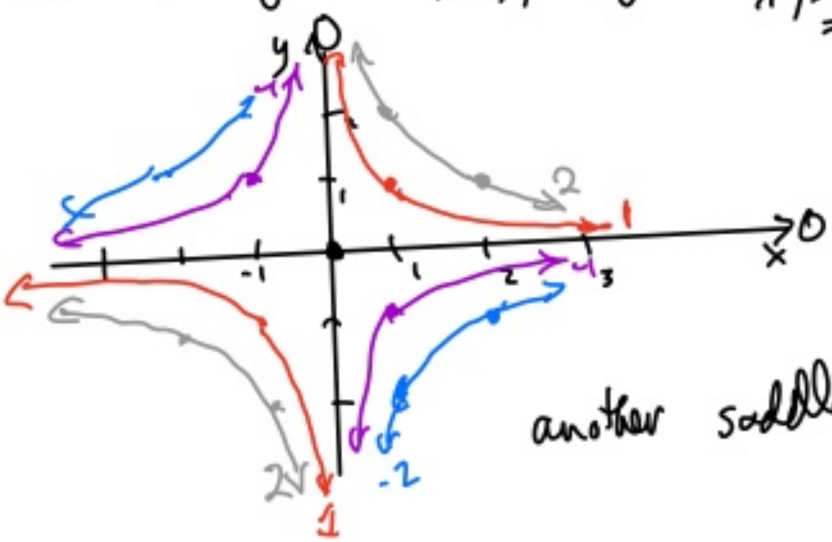
$2x^2 - 8y^2 = 0$
$x^2 - 4y^2 = 0$
$x^2 = 4y^2$
$y^2 = \frac{x^2}{4}$
$y = \pm \frac{x}{2}$

Another saddle shape.

② $z = xy$ $\quad f(x,y) = xy$

$xy = 0$
$\Rightarrow x = 0$
or $y = 0$

$xy = 1$
$y = \frac{1}{x}$



another saddle.

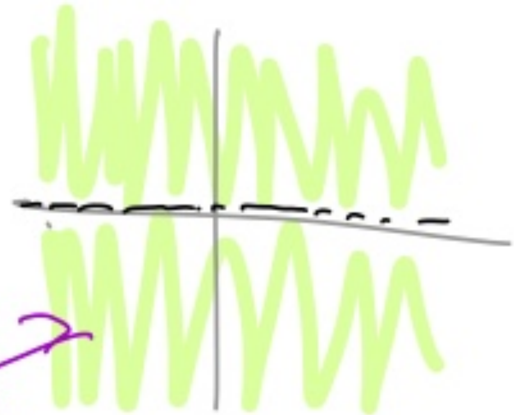
Domains of functions

$g(x) = \sqrt{3 - x^2}$ $\quad$ Domain of g = set of possible x-values for the function.

Domain$(g) = [-\sqrt{3}, +\sqrt{3}]$

Domain of $\left(h(x,y) = \frac{x}{y} \right)$.

$\text{Domain}(h) = \{ (x,y) : y \neq 0 \}$.
$= \mathbb{R}^2 \setminus \{x\text{-axis}\}$



Domain of h.

Domain of $F(x,y) = \sqrt{x^2 - 12 + y^2}$.

Can't have negative under square root.
need $x^2 + y^2 \geq 12$

$\text{Domain} = \{ (x,y) : x^2 + y^2 \geq 12 \}$

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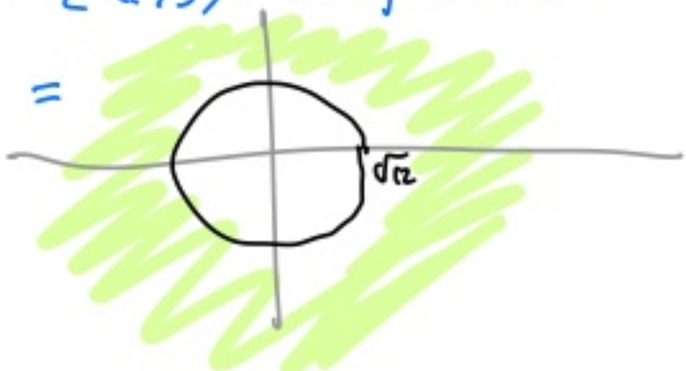


Limits in higher dimensions

Review: $\lim\limits_{x \to 0} \frac{\sin(3x)}{x^2 + x}$. $\overset{l'H}{=} \lim\limits_{x \to 0} \frac{3\cos(3x)}{2x+1} = \lim\limits_{x \to 0} \frac{3}{1} = \boxed{3}$.

$\frac{0}{0}$ form

$\boxed{\text{OR}}$ $\lim\limits_{x \to 0} \frac{\sin(3x)}{x^2 + x} = \lim\limits_{x \to 0} \frac{\sin(3x) \cdot 3}{3x(x+1)} = \lim\limits_{x \to 0} 1 \cdot \frac{3}{x+1} = \boxed{3}$

1

Basic idea $\lim\limits_{x \to a} f(x) = L \iff$ As x gets closer & closer to a (but not $= a$), then $f(x)$ gets closer & closer to L.

Higher dimensions

$$\lim_{(x,y,\dots)\to(a,b,\dots)} f(x,y,\dots) = L$$

$\iff$ As $(x,y,\dots)$ gets closer & closer to $(a,b,\dots)$ (but not equal), then $f(x,y,\dots)$ gets closer & closer to L.

Examples:

① $$\lim_{(x,y)\to(2,1)} \frac{x^2y+x}{2y^2} = \frac{2^2\cdot 1+2}{2\cdot 1^2} = \frac{6}{2} = \boxed{3}.$$

② $$\lim_{(x,y)\to(0,0)} \frac{xy^3+3x^2\sin(y)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2} + \frac{3x^2\sin(y)}{x^2+y^2}$$

Useful tool

$$x^2+y^2 \geq x^2 \Rightarrow 1 \geq \frac{x^2}{x^2+y^2}$$
$$x^2+y^2 \geq y^2 \Rightarrow 1 \geq \frac{y^2}{x^2+y^2}$$

Taking square roots:

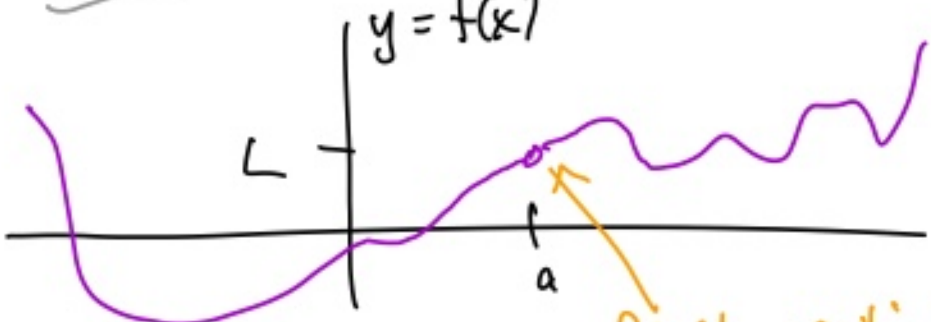
$$\frac{|x|}{\sqrt{x^2+y^2}} \leq 1$$
$$\frac{|y|}{\sqrt{x^2+y^2}} \leq 1.$$

$$x^2+y^2 = \sqrt{x^2+y^2}\,\sqrt{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\substack{\text{bounded}\\ -1\leq \blacksquare \leq 1}} \cdot \underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{\text{same}} \cdot \underbrace{y^2}_{\to 0} + 3\cdot \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\text{same}} \cdot \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\text{same}} \cdot \underbrace{\sin(y)}_{\to 0}$$

$= \boxed{0}$.

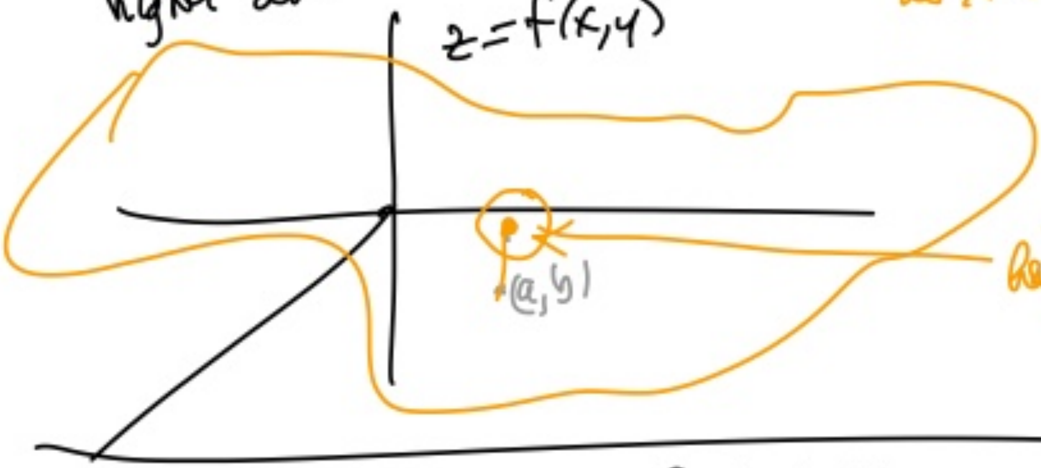
Note: we can see limits by graphing.



$\lim_{x \to a} f(x) = L$

limit = vertical position as $x \to a$

higher dim



height = limit.

③ $$\lim_{(x,y)\to(0,0)} \underbrace{\frac{xy + 3x^2\sin(y)}{x^2+y^2}}_{f(x,y)}$$

Thinking:

$$\underbrace{\underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\substack{\text{bdd} \\ -1 \le \bullet \le 1}} \cdot \underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{\text{same}}}_{} + \underbrace{3 \cdot \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\text{bdd}} \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\text{bdd}} \underbrace{\sin(y)}_{\substack{\downarrow \\ 0}}}_{\substack{\downarrow \\ 0}}$$

guess: limit does not exist (but bounded)

Let show the limit does not exist:

Along the line $x = 0$

$$f(x,y) = \frac{0}{0+y^2} = 0 \to 0 \text{ as } (x,y) \to (0,0).$$

Along the line $y=x$,

$$f(x,y) = \frac{x^2 + 3x^2\sin(x)}{x^2+x^2} = \frac{\cancel{x^2}(1+3\sin(x))}{2\cancel{x^2}}$$

$$\xrightarrow[x\to 0]{} \frac{1}{2} \neq 0$$

$\therefore$ limit does not exist.

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(4) $$\lim_{(x,y)\to(1,2)} \frac{x^2y - 2x^2 + y - 2}{y-2}$$ $\frac{0}{0}$ form.

$$= \lim_{(x,y)\to(1,2)} \frac{\cancel{(y-2)}(x^2+1)}{\cancel{(y-2)}} = \boxed{2}.$$

Recall from calc 1

$$\frac{df}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

= instantaneous rate of change of f as x increases

$f'(a)$ = slope of tangent line to $y=f(x)$ at $x=a$.

Several variables: Partial Derivatives

Example $F(x,y)$ is a function

$\frac{\partial F(x,y)}{\partial y}$ = partial derivative of F wrt y.

$$\frac{\partial F(x,y)}{\partial y} = \lim_{h \to 0} \frac{F(x,y+h) - F(x,y)}{h}$$

= derivative of $F(x,y)$ wrt y where x is fixed.

= instantaneous rate of change of $F(x,y)$ as y increases (and x stays constant)

= slope of the curve $z = F(x,y)$ with x fixed. at (x,y).

$$= \text{slope of } \left(\begin{matrix}\text{surface}\\ z = F(x,y)\end{matrix}\right) \cap \left(x = \text{const}\right)$$

Computing partial derivatives: use Calc 1 formulas, pretend all the other variables are constant.

eg. (1) $\frac{\partial}{\partial x}\left(e^{2xy} + x^2y^2\right)$

$$= 2ye^{2xy} + 2xy^2$$

(2) $\frac{\partial}{\partial y}\left(e^{2xy} + x^2y^2\right)$

$$= 2xe^{2xy} + 2x^2y$$

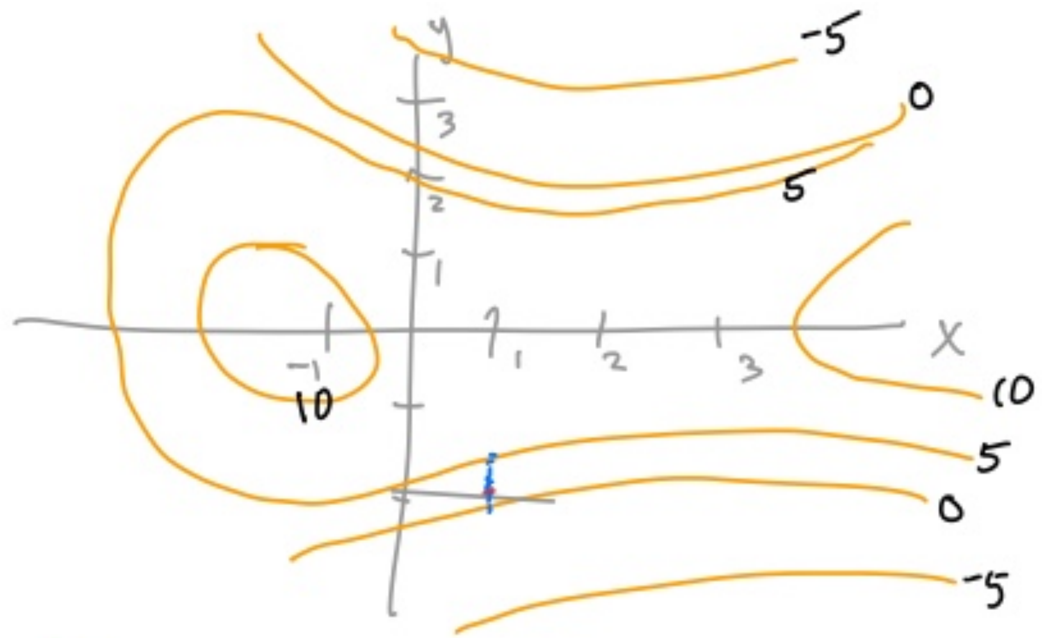
③ $\frac{\partial}{\partial y}\left(\sin\left(\frac{x}{y}\right)\ln(x+2y)\right)$

$= \frac{\partial}{\partial y}\left(\sin\left(\frac{x}{y}\right)\right)\cdot\ln(x+2y) + \sin\left(\frac{x}{y}\right)\frac{\partial}{\partial y}\left(\ln(x+2y)\right)$

$= \boxed{\cos\left(\frac{x}{y}\right)\cdot\left(\frac{-x}{y^2}\right)\cdot\ln(x+2y) + \sin\left(\frac{x}{y}\right)\left(\frac{1}{x+2y}\right)\cdot 2}$

Meaning: rates of change / slope.

Example Suppose $G(x,y)$ is given by this contour diagram:



Questions: ⓐ Estimate $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial y}$ at $(1,-2)$

$\frac{\partial G}{\partial x} \approx \underset{\text{small }\Delta x}{\frac{\Delta G}{\Delta x}} \approx \frac{\overset{0-5}{-5}}{1.3} \approx \boxed{-4}$

$\frac{\partial G}{\partial y} \approx \frac{\Delta G}{\Delta y} \approx \frac{\overset{5-0}{5}}{.684} \approx \boxed{7}$